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On a Discrete Korovkin Theorem

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In [G. A. Anastassiou, A discrete Korovkin theorem, *J. Approx. Theory* **45** (1985), pp. 383–388, Theorem 3], a discrete Korovkin theorem was given. We restate the theorem here and its proof, correcting a mistake in the above reference.

RESULTS

Our main result is the following

THEOREM 1. *Let $X = \{x_1, ..., x_j, ...\}$ be a set. Consider $B(X)$, the space of real valued bounded functions on X with the supremum norm $\|\cdot\|_\infty$, and a sequence of positive linear operators $L_n: B(X) \to B(X)$ such that $L_n(1, x_j) = 1$ for all j. Suppose that, for some $f_1, ..., f_k \in B(X)$,*

$$\lim_{n \to \infty} L_n(f_i, x_j) = f_i(x_j) \qquad \text{for all } i \text{ and } j. \tag{1.1}$$

In order that $L_n(f, x_j) \to f(x_j)$ for all $f \in B(X)$ and all j, it is enough to assume that for each j there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_j)) \geqslant 1 \qquad \text{for all} \quad x \in X - \{x_j\}. \tag{1.2}$$

We need the following

LEMMA 2. *Let $X = \{x_1, ..., x_j, ...\}$ be a measurable space, $1 \leqslant p < \infty$, and let μ be a finite positive measure on X such that $\mu(\{x_j\}) > 0$ for all j. Let $B(X)$ be as above, and let $f, f_1, f_2, ... \in B(X)$, where all $\|f_n\|_\infty < c$, $0 < c < \infty$. Then $f_n \to f$ pointwise on X iff $f_n \to f$ in $L_p(X, \mu)$.*

Proof of Lemma 2. ($\Rightarrow$) By the uniform boundedness of f_n, as $\mu(X) < \infty$, we obtain $|f_n|^p \leqslant c^p \in L_1(X, \mu)$. Since $f_n \to f$ pointwise, by a variation of the dominated convergence theorem (see [3, p. 180]), we get $f_n \to f$ in $L_p(X, \mu)$. Note $B(X) \subseteq L_p(X, \mu)$.

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($\Leftarrow$) The L_p convergence implies weak convergence, the indicator function $I_{\{x_j\}} \in L_q(X, \mu)$ where $(1/p)+(1/q)=1$, and $\mu(\{x_j\})>0$. Hence the pointwise convergence.

Next is an independent L_p result which will be used in the proof of Theorem 1.

PROPOSITION 3. *Let $X=\{x_1, ..., x_j, ...\}$ be a set. Let $w(x_j)>0$ for all j and $\sum_{j=1}^{\infty} w(x_j)<\infty$. Let $B(X)$ be as above, and let L_n be a sequence of positive linear operators: $B(X) \to B(X)$ such that $L_n(1, x_j)=1$ for all j. Suppose that, for some $f_1, f_2, ..., f_k \in B(X)$ and some p, $1 \leqslant p < \infty$,*

$$\lim_{n \to \infty} \left(\sum_{j=1}^{\infty} |L_n(f_i, x_j)-f_i(x_j)|^p \cdot w(x_j) \right)=0, \qquad i=1, 2, ..., k. \tag{3.1}$$

In order that $\sum_{j=1}^{\infty} |L_n(f, x_j)-f(x_j)|^p \cdot w(x_j) \to 0$ for all $f \in B(X)$, it is enough to assume the following: for each j there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x)-f_i(x_j)) \geqslant 1 \qquad \textit{for all} \quad x \in X-\{x_j\}. \tag{3.2}$$

Proof of Proposition 3. The weight w gives rise to a positive finite measure μ on X with $\mu(\{x\})>0$ for all $x \in X$. Since $B(X) \subseteq L_p(X, \mu)$, (3.1) implies $\|L_n(f_i)-f_i\|_p \to 0$ for all i. If there exists $f \in B(X)$ such that $\|L_n(f)-f\|_p \nrightarrow 0$, then there is j and a positive ε so that

$$|L_n(f, x_j)-f(x_j)|>\varepsilon \qquad \text{for all} \quad n \geqslant \text{some } n_0.$$

Because each positive linear functional $L_n(\cdot, x_j)$ on $B(X)$ is bounded, by a basic representation theorem, for each specific $j=j_0$ as above, there exists $g_{j_0,n} \in L_q(X, \mu)$, where $(1/p)+(1/q)=1$, such that

$$L_n(f, x_{j_0})=\int_X f(x) \cdot g_{j_0,n}(x) \cdot \mu(dx) \qquad \text{for all} \quad f \in B(X).$$

As $L_n(1, x_{j_0})=1$, the positivity of $L_n(\cdot, x_{j_0})$ implies $\int_X g_{j_0,n}(x) \cdot \mu(dx)=1$ and $g_{j_0,n}(x) \geqslant 0$ for all $x \in X$. Thus

$$\begin{aligned} \varepsilon &< |L_n(f, x_{j_0})-f(x_{j_0})| \\ &= \left| \int_{X-\{x_{j0}\}} (f(x)-f(x_{j_0})) \cdot g_{j_0,n}(x) \cdot \mu(dx) \right| \\ &\leqslant \|f-f(x_{j_0})\|_\infty \cdot \left(\int_{X-\{x_{j_0}\}} \cdot g_{j_0,n}(x) \cdot \mu(dx) \right), \end{aligned}$$

so

$$\int_{X-\{x_{j0}\}} g_{j_0,n}(x)\cdot\mu(dx) > \frac{\varepsilon}{\|f-f(x_{j0})\|_\infty} =: \delta > 0 \qquad \text{for all} \quad n \geqslant n_0.$$

There cannot be real constants $\beta_1, \beta_2, ..., \beta_k$ with

$$\sum_{i=1}^{k} \beta_i \cdot (f_i(x) - f_i(x_{j_0})) \geqslant 1 \qquad \text{for all} \quad x \in X - \{x_{j0}\}.$$

Since, otherwise, we would have

$$\sum_{i=1}^{k} \beta_i \cdot (f_i(x) - f_i(x_{j_0})) \cdot g_{j_0,n}(x) \geqslant g_{j_0,n}(x),$$

for all $x \in X - \{x_{j_0}\}$, and therefore

$$\sum_{i=1}^{k} \beta_i \cdot \int_{X-\{x_{j0}\}} (f_i(x) - f_i(x_{j_0})) \cdot g_{j_0,n}(x) \cdot \mu(dx)$$

$$\geqslant \int_{X-\{x_{j0}\}} g_{j_0,n}(x) \cdot \mu(dx).$$

(Note that $L_n(f_i, x_{j_0}) = \int_X f_i(x) \cdot g_{j_0,n}(x) \cdot \mu(dx)$, $i = 1, ..., k$.) Consequently, since $L_n(f_i, x_{j_0}) \to f_i(x_{j_0})$, $i = 1, ..., k$, we would get

$$0 = \lim_{n\to\infty} \left(\sum_{i=1}^{k} \beta_i \cdot (L_n(f_i, x_{j_0}) - f_i(x_{j_0})) \right) > \delta > 0.$$

Proof of Theorem 1. Note that $B(X) \subseteq L_p(X, \mu)$ for every $p \in [1, \infty)$, and every finite measure μ on X for which each $\mu(\{x_j\}) > 0$. By Lemma 2, the pointwise convergence $L_n(f_i, x_j) \to f_i(x_j)$, $i = 1, ..., k$, for all j, is equivalent, for such p and μ, to the convergence in $L_p(X, \mu)$ of $L_n(f_i)$ to f_i, $i = 1, ..., k$. Furthermore, this measure μ can serve as a weight function on X. Thus Proposition 3 implies our theorem.

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